SIMPLE EXTENSIONS OF REFLECTION SUBGROUPS OF PRIMITIVE COMPLEX REFLECTION GROUPS

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ABSTRACT. If G is a finite primitive complex reflection group, all reflection subgroups of G and their inclusions are determined up to conjugacy. As a consequence, it is shown that if the rank of G is n and if G can be generated by n reflections, then for every set R of n reflections which generate G, every subset of R generates a parabolic subgroup of G.

1. INTRODUCTION

The finite irreducible complex reflection groups were classified by Shephard and Todd [6] in 1954. If G is a primitive complex reflection group then, as shown by Shephard and Todd, G is either cyclic, a symmetric group $\mathrm{Sym}(n)$ for $n \geq 5$, or one of 34 groups G_k, where $4 \leq k \leq 37$.

A *reflection subgroup* of G is a subgroup generated by reflections. A *parabolic subgroup* is the pointwise stabiliser of a subset X of V. By a fundamental theorem of Steinberg [7] (see also [5, Theorem 9.44]) a parabolic subgroup is a reflection subgroup.

If H is a reflection subgroup of G, the *simple extensions* of H are the subgroups $\langle H, r\rangle$, where r is a reflection and $r \notin H$.

If $\mathcal{H}$ is a conjugacy class of reflection subgroups of G, a conjugacy class $\mathcal{K}$ is a *simple extension* of $\mathcal{H}$ if there exists $H \in \mathcal{H}$ and $K \in \mathcal{K}$ such that K is a simple extension of H.

All simple extensions of the conjugacy classes of reflection subgroups of the imprimitive complex reflection groups $G(m, p, n)$ were determined in [8]. The purpose of the present paper is to extend this result to all finite complex reflection groups by describing the simple extensions of the conjugacy classes of reflection subgroups of the groups G_k $(23 \leq k \leq 37)$. For the groups G_k of rank 2 $(4 \leq k \leq 22)$ every element is a reflection modulo scalars and the simple extensions can be deduced from the results of [5, Chapter 6].

The results are presented in Section 5 in the form of tables. The tables themselves were computed with the aid of the computational algebra system MAGMA [1]. Tables of conjugacy classes of the reflection subgroups of the Coxeter groups of types E_6, E_7, E_8, F_4, H_3 and H_4 can be found in [3]. Tables of conjugacy classes of parabolic subgroups of these groups also appear in [4, Appendix A].

Refer to [5] and [8] for background and terminology not otherwise explained here.

2. NOTATION

In his thesis Cohen [2] introduced a notation for primitive complex reflection groups of rank at least 3 which extends the standard (Cartan) notation for Coxeter groups. In this notation the complex reflection groups which are not Coxeter groups are labelled $J_3^{(4)}$, $J_3^{(5)}$, K_5, K_6,

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L_3, L_4, M_3, N_4 and EN_4. In the tables which follow we shall label the conjugacy classes of reflection subgroups using this notation except that, as in [5], we use O_4 instead of EN_4.

A reflection subgroup which is the direct product of irreducible reflection groups of types $T_1, T_2, \ldots, T_k$ will be labelled $T_1 + T_2 + \cdots + T_k$ and if $T_i = T$ for all i we denote the group by kT.

For the imprimitive reflection subgroups which occur in the tables we use the notation introduced in [5, section 7.5] rather than the Shephard and Todd notation $G(m, p, n)$. That is, $B_n^{(2p)}$ denotes the group $G(2p, p, n)$ and $D_n^{(p)}$ denotes the group $G(p, p, n)$. For consistency with the Cartan names we write B_n instead of $B_n^{(2)}$ and D_n instead of $D_n^{(2)}$. Similarly A_{n-1} denotes the symmetric group $\mathrm{Sym}(n) \simeq G(1, 1, n)$. However, we use $D_2^{(m)}$ rather than $I_2(m)$ to denote the dihedral group of order $2m$.

For small values of the parameters there are isomorphisms between the groups: $B_2 \simeq D_2^{(4)}$, $A_2 \simeq D_2^{(3)}$, $A_3 \simeq D_3$ and $B_2 \simeq D_2^{(4)}$. The tables use the first named symbol for these groups. The cyclic groups of order 2 and 3 are denoted by A_1 and L_1 respectively, and L_2 denotes the Shephard and Todd group G_4.

If there is more than one conjugacy class of reflection subgroups of type T we label the conjugacy classes $T.1$, $T.2$, and so on. There is no significance to the order in which these indices occur.

Given a reflection subgroup H, the *parabolic closure* of H is the pointwise stabiliser of the space of fixed points of H; it is the smallest parabolic subgroup which contains H. The rank of the parabolic closure equals is equal to the rank of H. For those (conjugacy classes of) reflection subgroups H whose parabolic closure is a simple extension of H we place the parabolic closure first in the list of simple extensions and use a bold font.

The conjugacy classes of parabolic subgroups are labelled with the symbol $\wp$.

3. Main theorem

Theorem 3.1. *Suppose that H is a reflection subgroup of the finite primitive complex reflection group G and suppose that K is a simple extension of H. If K is parabolic and the rank of K is greater than the rank of H, then H is parabolic.*

Proof. If G is the symmetric group $\mathrm{Sym}(n)$, it is well known—see [8, Corollary 3.9] for a proof—that every reflection subgroup of G is parabolic. Thus in this case there is nothing to prove. If the rank of G is 1, the only parabolic subgroup is G itself and so we may suppose that $G = G_k$ for some k such that $4 \le k \le 37$.

If the rank of G is 2 and H is a non-parabolic reflection subgroup of rank 1, then from [5, Table D.1] G contains an element of order 4 whose square generates H. If H has a simple extension of rank 2 which is parabolic, then the simple extension is G and so G is generated by two reflections. The only possibilities for G are G_8 and G_9 but from [5, Section 6.3] neither group can be generated by two reflections one of which is the square of a reflection of order 4.

If the rank of G is at least 3, the result follows from an inspection of the tables in Section 5. □

Corollary 3.2. *If G is a primitive reflection group of rank n and if R is a set of n reflections which generate G, then for any subset S of R, the subgroup generated by S is a parabolic subgroup of G.*

4. The Magma code

In order to construct the complex reflection group $W = G_n$, where $4 \le n \le 37$, use the Magma code

```
roots, coroots, rho, W, J := ComplexRootDatum(n);
```

In addition to W this function returns a set of roots, a set of coroots, a bijection `rho` from `roots` to `coroots` and a matrix J which defines a W-invariant positive definite hermitian form. The reflection `r` with root `a` and coroot `rho(a)` can be obtained via the code

```
r := PseudoReflection(a,rho(a));
```

Given a reflection subgroup H of W, we create a sequence `extn` of simple extensions of H (up to conjugacy):

(1) Let `orbreps` be a set of representatives for the orbits of the normaliser of H in W on the reflections which do not belong to H.

(2) For each reflection `r` in `orbreps` construct the simple extension

```
G := sub< W | H, r >;
```

(3) If G is not conjugate in W to any simple extension already constructed, append G to `extn`.

Identification of the type of a reflection subgroup H is carried out as follows.

(1) Compute the list of indecomposable components $L_1, L_2, \ldots, L_k$ of the root system L of H; that is, L is the union of the L_i, the L_i are pairwise orthogonal and the reflection subgroup of H corresponding to L_i is irreducible (as a reflection group).

(2) Compute the standard name of each indecomposable component of H. This is facilitated by the observation that the irreducible reflection groups K which occur in the tables are uniquely determined by the pair of integers (n, m), where n is the order of K and m is the size of its line system.

(3) An associative array `refgroup` is used to map the standard name of a reflection subgroup to the actual subgroup.

```
refgroup := AssociativeArray(Parent(""));
```

The full implementation of the Magma code is available at

http://www.maths.usyd.edu.au/u/don/

in the file `subsystems.m`. The function `showTable` creates the data which is the basis for the tables in Section 5. For example

```
load "subsystems.m";
showTable(23);
```

displays the data

```
P | A1 | [ A1A1, A2, D2(5) ]
P | A1A1 | [ H3, A1A1A1 ]
P | A2 | [ H3 ]
P | D2(5) | [ H3 ]
P | H3 | []
N | A1A1A1 | [ H3 ]
```

5. The tables

Table 1. Reflection subgroup classes of $G_{23} = H_3$

	Class	Simple extensions
$\wp$	A_1	$D_2^{(5)},\ A_2,\ 2A_1$
$\wp$	$2A_1$	$H_3,\ 3A_1$
$\wp$	A_2	H_3
$\wp$	$D_2^{(5)}$	H_3
	$3A_1$	$\mathbf{H}_3$

Table 2. Reflection subgroup classes of $G_{24} = J_3^{(4)}$

	Class	Simple extensions
$\wp$	A_1	$B_2,\ A_2,\ 2A_1.1,\ 2A_1.2$
	$2A_1.1$	$\mathbf{B}_2,\ B_3.1,\ A_1+B_2,\ A_3.1,\ 3A_1.1$
	$2A_1.2$	$\mathbf{B}_2,\ B_3.2,\ A_1+B_2,\ A_3.2,\ 3A_1.2$
$\wp$	A_2	$J_3^{(4)},\ B_3.1,\ B_3.2,\ A_3.1,\ A_3.2$
$\wp$	B_2	$J_3^{(4)},\ B_3.1,\ B_3.2,\ A_1+B_2$
	$3A_1.1$	$B_3.1,\ A_1+B_2$
	$3A_1.2$	$B_3.2,\ A_1+B_2$
	A_1+B_2	$\mathbf{J}_3^{(4)},\ B_3.1,\ B_3.2$
	$A_3.1$	$\mathbf{J}_3^{(4)},\ B_3.1$
	$A_3.2$	$\mathbf{J}_3^{(4)},\ B_3.2$
	$B_3.1$	$\mathbf{J}_3^{(4)}$
	$B_3.2$	$\mathbf{J}_3^{(4)}$

Table 3. Reflection subgroup classes of $G_{25} = L_3$

	Class	Simple extensions
$\wp$	L_1	$L_2,\ 2L_1$
$\wp$	$2L_1$	$L_3,\ 3L_1$
$\wp$	L_2	L_3
	$3L_1$	$\mathbf{L}_3$

Table 4. Reflection subgroup classes of $G_{26} = M_3$

	Class	Simple extensions
$\wp$	L_1	$L_2,\ 2L_1,\ B_2^{(3)},\ A_1+L_1$
$\wp$	A_1	$B_2^{(3)},\ A_1+L_1,\ A_2$
	$2L_1$	$\mathbf{B}_2^{(3)},\ L_3,\ B_2^{(3)}+L_1,\ 3L_1$
	A_2	$\mathbf{B}_2^{(3)},\ B_3^{(3)},\ D_3^{(3)},\ A_2+L_1$
$\wp$	L_2	$M_3,\ L_3,\ A_1+L_2$
$\wp$	$B_2^{(3)}$	$M_3,\ B_3^{(3)},\ B_2^{(3)}+L_1$
$\wp$	A_1+L_1	$M_3,\ B_3^{(3)},\ B_2^{(3)}+L_1,\ A_1+L_2,\ A_2+L_1$
	$D_3^{(3)}$	$B_3^{(3)}$
	$3L_1$	$L_3,\ B_2^{(3)}+L_1$
	$B_2^{(3)}+L_1$	$\mathbf{M}_3,\ B_3^{(3)}$
	A_2+L_1	$\mathbf{M}_3,\ B_3^{(3)},\ B_2^{(3)}+L_1$
	$B_3^{(3)}$	$\mathbf{M}_3$
	L_3	$\mathbf{M}_3$
	A_1+L_2	$\mathbf{M}_3$

Table 5. Reflection subgroup classes of $G_{27} = J_3^{(5)}$

	Class	Simple extensions
$\wp$	A_1	$B_2,\ D_2^{(5)},\ A_2.1,\ A_2.2,\ 2A_1.1,\ 2A_1.2$
	$2A_1.1$	$\mathbf{B}_2,\ H_3.1,\ B_3.1,\ A_1+B_2,\ A_3.1,\ 3A_1.1$
	$2A_1.2$	$\mathbf{B}_2,\ H_3.2,\ B_3.2,\ A_1+B_2,\ A_3.2,\ 3A_1.2$
$\wp$	$A_2.1$	$J_3^{(5)},\ H_3.2,\ B_3.1,\ D_3^{(3)},\ A_3.1$
$\wp$	$A_2.2$	$J_3^{(5)},\ H_3.1,\ B_3.2,\ D_3^{(3)},\ A_3.2$
$\wp$	$D_2^{(5)}$	$J_3^{(5)},\ H_3.1,\ H_3.2$
$\wp$	B_2	$J_3^{(5)},\ B_3.1,\ B_3.2,\ A_1+B_2$
	$3A_1.1$	$H_3.1,\ B_3.1,\ A_1+B_2$
	$3A_1.2$	$H_3.2,\ B_3.2,\ A_1+B_2$
	$A_3.1$	$\mathbf{J}_3^{(5)},\ B_3.1$
	$A_3.2$	$\mathbf{J}_3^{(5)},\ B_3.2$
	A_1+B_2	$\mathbf{J}_3^{(5)},\ B_3.1,\ B_3.2$
	$H_3.1$	$\mathbf{J}_3^{(5)}$
	$H_3.2$	$\mathbf{J}_3^{(5)}$
	$B_3.1$	$\mathbf{J}_3^{(5)}$
	$B_3.2$	$\mathbf{J}_3^{(5)}$
	$D_3^{(3)}$	$\mathbf{J}_3^{(5)}$

Table 6. Reflection subgroup classes of $G_{28} = F_4$

	Class	Simple extensions
$\wp$	$A_1.1$	$B_2,\ A_2.1,\ 2A_1.1,\ 2A_1.3$
$\wp$	$A_1.2$	$B_2,\ A_2.2,\ 2A_1.2,\ 2A_1.3$
	$2A_1.1$	$\mathbf{B}_2,\ (A_1+B_2).2,\ A_3.1,\ 3A_1.1,\ 3A_1.2$
	$2A_1.2$	$\mathbf{B}_2,\ (A_1+B_2).1,\ A_3.2,\ 3A_1.3,\ 3A_1.4$
$\wp$	$2A_1.3$	$B_3.1,\ B_3.2,\ (A_1+B_2).2,\ (A_1+B_2).1,$ $(A_1+A_2).1,\ (A_1+A_2).2,\ 3A_1.2,\ 3A_1.3$
$\wp$	$A_2.1$	$B_3.1,\ A_3.1,\ (A_1+A_2).1$
$\wp$	$A_2.2$	$B_3.2,\ A_3.2,\ (A_1+A_2).2$
$\wp$	B_2	$B_3.1,\ B_3.2,\ (A_1+B_2).1,\ (A_1+B_2).2$
	$A_3.1$	$\mathbf{B}_3.1,\ B_4.1,\ D_4.1,\ (A_1+A_3).1$
	$A_3.2$	$\mathbf{B}_3.2,\ B_4.2,\ D_4.2,\ (A_1+A_3).2$
	$3A_1.1$	$(\mathbf{A}_1+\mathbf{B}_2).2,\ (2A_1+B_2).1,\ D_4.1,\ 4A_1.1$
	$3A_1.2$	$\mathbf{B}_3.1,\ (A_1+B_3).2,\ (A_1+B_2).1,\ (2A_1+B_2).1,$ $(A_1+A_3).1,\ 4A_1.2$
	$3A_1.3$	$\mathbf{B}_3.2,\ (A_1+B_3).1,\ (A_1+B_2).2,\ (2A_1+B_2).2,$ $(A_1+A_3).2,\ 4A_1.2$
	$3A_1.4$	$(\mathbf{A}_1+\mathbf{B}_2).1,\ (2A_1+B_2).2,\ D_4.2,\ 4A_1.3$
	$(A_1+B_2).1$	$\mathbf{B}_3.1,\ B_4.2,\ 2B_2,\ (A_1+B_3).1,\ (2A_1+B_2).2$
	$(A_1+B_2).2$	$\mathbf{B}_3.2,\ B_4.1,\ 2B_2,\ (A_1+B_3).2,\ (2A_1+B_2).1$
$\wp$	$(A_1+A_2).1$	$F_4,\ B_4.1,\ (A_1+B_3).1,\ 2A_2,\ (A_1+A_3).1$
$\wp$	$(A_1+A_2).2$	$F_4,\ B_4.2,\ (A_1+B_3).2,\ 2A_2,\ (A_1+A_3).2$
$\wp$	$B_3.1$	$F_4,\ B_4.1,\ (A_1+B_3).1$
$\wp$	$B_3.2$	$F_4,\ B_4.2,\ (A_1+B_3).2$
	$4A_1.1$	$D_4.1,\ (2A_1+B_2).1$
	$4A_1.2$	$(2A_1+B_2).1,\ (2A_1+B_2).2,\ (A_1+B_3).1,\ (A_1+B_3).2$
	$4A_1.3$	$D_4.2,\ (2A_1+B_2).2$
	$D_4.1$	$B_4.1$
	$D_4.2$	$B_4.2$
	$(2A_1+B_2).1$	$2B_2,\ B_4.1,\ (A_1+B_3).2$
	$(2A_1+B_2).2$	$2B_2,\ B_4.2,\ (A_1+B_3).1$
	$2B_2$	$B_4.1,\ B_4.2$
	$(A_1+A_3).1$	$\mathbf{F}_4,\ B_4.1,\ (A_1+B_3).1$
	$(A_1+A_3).2$	$\mathbf{F}_4,\ B_4.2,\ (A_1+B_3).2$
	$(A_1+B_3).1$	$\mathbf{F}_4,\ B_4.1$
	$(A_1+B_3).2$	$\mathbf{F}_4,\ B_4.2$
	$B_4.1$	$\mathbf{F}_4$
	$B_4.2$	$\mathbf{F}_4$
	$2A_2$	$\mathbf{F}_4$

Table 7. Reflection subgroup classes of $G_{29} = N_4$

	Class	Simple extensions
$\wp$	A_1	$B_2,\ A_2,\ 2A_1.1,\ 2A_1.2$
	$2A_1.1$	$\mathbf{B}_2,\ A_1+B_2,\ A_3.1,\ A_3.4,\ 3A_1.1,\ 3A_1.2$
$\wp$	$2A_1.2$	$B_3,\ A_1+B_2,\ A_1+A_2,\ A_3.2,\ A_3.3,\ 3A_1.2$
$\wp$	A_2	$B_3,\ D_3^{(4)},\ A_1+A_2,\ A_3.1,\ A_3.2,\ A_3.3,\ A_3.4$
$\wp$	B_2	$B_3,\ D_3^{(4)},\ A_1+B_2$
	$3A_1.1$	$A_1+B_2,\ 2A_1+B_2,\ D_4.1,\ 4A_1.1$
	$3A_1.2$	$\mathbf{B}_3,\ A_1+B_2,\ A_1+B_3,\ 2A_1+B_2,\ D_4.2,\ 4A_1.2,\ A_1+A_3$
	A_1+B_2	$\mathbf{B}_3,\ 2B_2,\ B_4,\ A_1+B_3,\ 2A_1+B_2,\ D_4^{(4)}$
	$A_3.1$	$\mathbf{B}_3,\ D_4.1,\ D_4^{(4)},\ A_1+A_3$
	$A_3.4$	$\mathbf{D}_3^{(4)},\ B_4,\ D_4.1,\ D_4.2,\ D_4^{(4)}$
$\wp$	$A_3.2$	$N_4,\ D_4.2,\ D_4^{(4)},\ A_4.1$
$\wp$	$A_3.3$	$N_4,\ D_4.2,\ D_4^{(4)},\ A_4.2$
$\wp$	A_1+A_2	$N_4,\ B_4,\ A_1+B_3,\ A_1+A_3,\ A_4.1,\ A_4.2$
$\wp$	B_3	$N_4,\ B_4,\ A_1+B_3$
$\wp$	$D_3^{(4)}$	$N_4,\ D_4^{(4)}$
	$4A_1.1$	$D_4.1,\ 2A_1+B_2$
	$4A_1.2$	$D_4.2,\ 2A_1+B_2,\ A_1+B_3$
	$2A_1+B_2$	$B_4,\ A_1+B_3,\ D_4^{(4)},\ 2B_2$
	$2B_2$	$B_4,\ D_4^{(4)}$
	$D_4.1$	$B_4,\ D_4^{(4)}$
	$D_4.2$	$\mathbf{N}_4,\ D_4^{(4)}$
	A_1+A_3	$\mathbf{N}_4,\ A_1+B_3,\ B_4$
	A_1+B_3	$\mathbf{N}_4,\ B_4$
	$A_4.1$	$\mathbf{N}_4$
	$A_4.2$	$\mathbf{N}_4$
	B_4	$\mathbf{N}_4$
	$D_4^{(4)}$	$\mathbf{N}_4$

TABLE 8. Reflection subgroup classes of $G_{30} = H_4$

	Class	Simple extensions
$\wp$	A_1	$D_2^{(5)},\ A_2,\ 2A_1$
$\wp$	$2A_1$	$H_3,\ A_1 + D_2^{(5)},\ A_1 + A_2,\ A_3,\ 3A_1$
$\wp$	A_2	$H_3,\ A_1 + A_2,\ A_3$
$\wp$	$D_2^{(5)}$	$H_3,\ A_1 + D_2^{(5)}$
	$3A_1$	$\mathbf{H}_3,\ A_1 + H_3,\ D_4,\ 4A_1$
$\wp$	A_3	$H_4,\ D_4,\ A_4$
$\wp$	$A_1 + A_2$	$H_4,\ A_1 + H_3,\ A_4,\ 2A_2$
$\wp$	$A_1 + D_2^{(5)}$	$H_4,\ A_1 + H_3,\ 2D_2^{(5)}$
$\wp$	H_3	$H_4,\ A_1 + H_3$
	$4A_1$	$A_1 + H_3,\ D_4$
	$A_1 + H_3$	$\mathbf{H}_4$
	D_4	$\mathbf{H}_4$
	$2D_2^{(5)}$	$\mathbf{H}_4$
	A_4	$\mathbf{H}_4$
	$2A_2$	$\mathbf{H}_4$

TABLE 9. Reflection subgroup classes of $G_{31} = O_4$

	Class	Simple extensions (ranks 1, 2 and 3)
$\wp$	A_1	$B_2,\ A_2,\ 2A_1.1,\ 2A_1.2$
	$2A_1.2$	$B_2,\ (A_1 + B_2).1,\ A_3.2,\ 3A_1.1,\ 3A_1.2$
	B_2	$\mathbf{B}_2^{(4)},\ B_3,\ D_3^{(4)},\ (A_1 + B_2).1,\ (A_1 + B_2).2$
$\wp$	$2A_1.1$	$B_3,\ (A_1 + B_2).1,\ (A_1 + B_2).2,\ A_1 + A_2,\ A_3.1,\ 3A_1.1$
$\wp$	A_2	$B_3,\ D_3^{(4)},\ A_1 + A_2,\ A_3.1,\ A_3.2$
$\wp$	$B_2^{(4)}$	$B_3^{(4)},\ A_1 + B_2^{(4)}$
	$A_3.2$	$B_3,\ D_3^{(4)},\ B_4.1,\ D_4^{(4)},\ D_4.1,\ D_4.2,\ A_1 + A_3$
	$3A_1.1$	$B_3,\ (A_1 + B_2).1,\ (A_1 + B_2).2,\ D_4.1,\ A_1 + B_3,$ $A_1 + A_3,\ (2A_1 + B_2).1,\ (2A_1 + B_2).2,\ 4A_1.1$
	$3A_1.2$	$(A_1 + B_2).1,\ D_4.2,\ (2A_1 + B_2).1,\ 4A_1.2$
	$(A_1 + B_2).1$	$B_3,\ A_1 + B_2^{(4)},\ B_4.1,\ 2B_2.1,\ 2B_2.2,\ D_4^{(4)},\ A_1 + B_3,$ $(2A_1 + B_2).1$
	$(A_1 + B_2).2$	$\mathbf{B}_3^{(4)},\ A_1 + B_2^{(4)},\ B_4.2,\ A_1 + D_3^{(4)},\ 2B_2.2,\ (2A_1 + B_2).2$
	$A_1 + B_2^{(4)}$	$\mathbf{B}_3^{(4)},\ B_4^{(4)},\ A_1 + B_3^{(4)},\ B_2 + B_2^{(4)},\ 2A_1 + B_2^{(4)}$
	B_3	$\mathbf{B}_3^{(4)},\ N_4,\ F_4,\ B_4.1,\ B_4.2,\ A_1 + B_3$
	$D_3^{(4)}$	$\mathbf{B}_3^{(4)},\ N_4,\ D_4^{(4)},\ A_1 + D_3^{(4)}$
$\wp$	$A_3.1$	$N_4,\ B_4.2,\ D_4^{(4)},\ D_4.1,\ A_4.1,\ A_4.2$
$\wp$	$A_1 + A_2$	$N_4,\ F_4,\ B_4.1,\ B_4.2,\ A_1 + B_3,\ A_1 + D_3^{(4)},\ A_1 + A_3,$ $2A_2,\ A_4.1,\ A_4.2$
$\wp$	$B_3^{(4)}$	$O_4,\ B_4^{(4)},\ A_1 + B_3^{(4)}$

Table 10. Reflection subgroup classes of $G_{31} = O_4$ (continued)

Class	Simple extensions (rank 4)
$4A_1.1$	$A_1 + B_3,\ (2A_1 + B_2).1,\ (2A_1 + B_2).2,\ D_4.1$
$4A_1.2$	$D_4.2,\ (2A_1 + B_2).1$
$(2A_1 + B_2).1$	$B_4.1,\ D_4^{(4)},\ A_1 + B_3,\ 2A_1 + B_2^{(4)},\ 2B_2.1,\ 2B_2.2$
$(2A_1 + B_2).2$	$A_1 + B_3^{(4)},\ 2B_2.2,\ B_4.2,\ 2A_1 + B_2^{(4)}$
$A_1 + A_3$	$N_4,\ F_4,\ B_4.1,\ B_4.2,\ A_1 + D_3^{(4)},\ A_1 + B_3$
$A_1 + B_3$	$N_4,\ F_4,\ A_1 + B_3^{(4)},\ B_4.1,\ B_4.2$
$2B_2.1$	$B_4.1,\ D_4^{(4)},\ B_2 + B_2^{(4)}$
$2B_2.2$	$B_4.2,\ B_4^{(4)},\ B_2 + B_2^{(4)}$
$2A_1 + B_2^{(4)}$	$B_4^{(4)},\ A_1 + B_3^{(4)},\ B_2 + B_2^{(4)}$
$B_2 + B_2^{(4)}$	$B_4^{(4)},\ 2B_2^{(4)}$
$D_4.1$	$N_4,\ B_4.2,\ D_4^{(4)}$
$D_4.2$	$B_4.1,\ D_4^{(4)}$
$2B_2^{(4)}$	$B_4^{(4)}$
$D_4^{(4)}$	$N_4,\ B_4^{(4)}$
$B_4.1$	$N_4,\ B_4^{(4)},\ F_4$
$B_4.2$	$\mathbf{O}_4,\ B_4^{(4)}$
$A_1 + D_3^{(4)}$	$\mathbf{O}_4,\ B_4^{(4)},\ A_1 + B_3^{(4)}$
$A_1 + B_3^{(4)}$	$\mathbf{O}_4,\ B_4^{(4)}$
$A_4.1$	$\mathbf{O}_4,\ N_4$
$A_4.2$	$\mathbf{O}_4,\ N_4$
$2A_2$	$\mathbf{O}_4,\ F_4$
$B_4^{(4)}$	$\mathbf{O}_4$
F_4	$\mathbf{O}_4$
N_4	$\mathbf{O}_4$

Table 11. Reflection subgroup classes of $G_{32} = L_4$

	Class	Simple extensions
$\wp$	L_1	$L_2,\ 2L_1$
$\wp$	$2L_1$	$L_3,\ L_1 + L_2,\ 3L_1$
$\wp$	L_2	$L_3,\ L_1 + L_2$
	$3L_1$	$\mathbf{L}_3,\ L_1 + L_3,\ 4L_1$
$\wp$	L_3	$L_4,\ L_1 + L_3$
$\wp$	$L_1 + L_2$	$L_4,\ L_1 + L_3,\ 2L_2$
	$4L_1$	$L_1 + L_3$
	$L_1 + L_3$	$\mathbf{L}_4$
	$2L_2$	$\mathbf{L}_4$

Table 12. Reflection subgroup classes of $G_{33} = K_5$

	Class	Simple extensions
$\wp$	A_1	$A_2,\ 2A_1$
$\wp$	$2A_1$	$A_1 + A_2,\ A_3,\ 3A_1$
$\wp$	A_2	$D_3^{(3)},\ A_1 + A_2,\ A_3$
$\wp$	$A_1 + A_2$	$D_4^{(3)},\ A_1 + A_3,\ A_4,\ 2A_2$
$\wp$	A_3	$D_4,\ D_4^{(3)},\ A_1 + A_3,\ A_4$
$\wp$	$3A_1$	$D_4,\ A_1 + A_3,\ 4A_1$
$\wp$	$D_3^{(3)}$	$D_4^{(3)}$
	$4A_1$	$\mathbf{D}_4,\ A_1 + D_4,\ 5A_1$
	$2A_2$	$\mathbf{D}_4^{(3)},\ A_5$
$\wp$	$A_1 + A_3$	$K_5,\ A_1 + D_4,\ A_5$
$\wp$	D_4	$K_5,\ A_1 + D_4$
$\wp$	A_4	$K_5,\ A_5$
$\wp$	$D_4^{(3)}$	K_5
	$5A_1$	$A_1 + D_4$
	$A_1 + D_4$	$\mathbf{K}_5$
	A_5	$\mathbf{K}_5$

Table 13. Reflection subgroup classes of $G_{34} = K_6$

	Class	Simple extensions (ranks 1 to 4)
$\wp$	A_1	$A_2,\ 2A_1$
$\wp$	$2A_1$	$A_1 + A_2,\ A_3,\ 3A_1$
$\wp$	A_2	$D_3^{(3)},\ A_1 + A_2,\ A_3$
$\wp$	$A_1 + A_2$	$D_4^{(3)},\ A_1 + D_3^{(3)},\ A_1 + A_3,\ 2A_1 + A_2,\ A_4,\ 2A_2.1,\ 2A_2.2$
$\wp$	A_3	$A_4,\ D_4,\ D_4^{(3)},\ A_1 + A_3$
$\wp$	$3A_1$	$D_4,\ A_1 + A_3,\ 2A_1 + A_2,\ 4A_1$
$\wp$	$D_3^{(3)}$	$D_4^{(3)},\ A_1 + D_3^{(3)}$
	$4A_1$	$\mathbf{D}_4,\ A_1 + D_4,\ 2A_1 + A_3,\ 5A_1$
	$2A_2.2$	$\mathbf{D}_4^{(3)},\ A_2 + D_3^{(3)},\ A_1 + 2A_2,\ A_5.2$
$\wp$	$2A_2.1$	$D_5^{(3)},\ A_2 + D_3^{(3)},\ A_2 + A_3,\ A_5.1,\ A_5.3$
$\wp$	$A_1 + A_3$	$K_5,\ D_5,\ A_1 + D_4^{(3)},\ A_1 + D_4,\ A_2 + A_3,\ A_1 + A_4,$ $2A_1 + A_3,\ A_5.1,\ A_5.2,\ A_5.3$
$\wp$	$2A_1 + A_2$	$D_5,\ A_1 + D_4^{(3)},\ A_2 + A_3,\ A_1 + A_4,\ A_1 + 2A_2,\ 2A_1 + A_3$
$\wp$	$A_1 + D_3^{(3)}$	$D_5^{(3)},\ A_2 + D_3^{(3)},\ A_1 + D_4^{(3)}$
$\wp$	D_4	$K_5,\ D_5,\ A_1 + D_4$
$\wp$	A_4	$K_5,\ D_5,\ D_5^{(3)},\ A_1 + A_4,\ A_5.1,\ A_5.2,\ A_5.3$
$\wp$	$D_4^{(3)}$	$K_5,\ D_5^{(3)},\ A_1 + D_4^{(3)}$

Table 14. Reflection subgroup classes of $G_{34} = K_6$ (continued)

	Class	Simple extensions (ranks 5 and 6)
	$5A_1$	$A_1 + D_4,\ 2A_1 + D_4,\ 6A_1$
	$A_2 + D_3^{(3)}$	$\mathbf{D}_5^{(3)},\ D_6^{(3)},\ 2D_3^{(3)},\ A_2 + D_4^{(3)}$
	$A_1 + D_4$	$\mathbf{K}_5,\ A_1 + K_5,\ D_6,\ 2A_1 + D_4$
	$2A_1 + A_3$	$\mathbf{D}_5,\ A_1 + K_5,\ D_6,\ 2A_1 + D_4,\ A_1 + A_5,\ 2A_3$
	$A_1 + 2A_2$	$\mathbf{A}_1 + \mathbf{D}_4^{(3)},\ E_6,\ A_2 + D_4^{(3)},\ A_1 + A_5,\ 3A_2$
	$A_5.2$	$\mathbf{K}_5,\ E_6,\ D_6^{(3)},\ A_1 + A_5$
$\wp$	$A_5.1$	$K_6,\ D_6,\ D_6^{(3)},\ A_6.1$
$\wp$	$A_5.3$	$K_6,\ D_6,\ D_6^{(3)},\ A_6.2$
$\wp$	$A_2 + A_3$	$K_6,\ D_6,\ D_6^{(3)},\ A_2 + D_4^{(3)},\ 2A_3,\ A_6.1,\ A_6.2$
$\wp$	$A_1 + A_4$	$K_6,\ E_6,\ A_1 + K_5,\ A_1 + A_5,\ A_6.1,\ A_6.2$
$\wp$	$A_1 + D_4^{(3)}$	$K_6,\ A_1 + K_5,\ D_6^{(3)},\ A_2 + D_4^{(3)}$
$\wp$	D_5	$K_6,\ E_6,\ D_6$
$\wp$	$D_5^{(3)}$	$K_6,\ D_6^{(3)}$
$\wp$	K_5	$K_6,\ A_1 + K_5$
	$6A_1$	$2A_1 + D_4$
	$2A_1 + D_4$	$A_1 + K_5,\ D_6$
	$3A_2$	$E_6,\ A_2 + D_4^{(3)}$
	$2D_3^{(3)}$	$D_6^{(3)}$
	$A_1 + A_5$	$\mathbf{K}_6,\ A_1 + K_5,\ E_6$
	$A_2 + D_4^{(3)}$	$\mathbf{K}_6,\ D_6^{(3)}$
	$2A_3$	$\mathbf{K}_6,\ D_6$
	$A_6.1$	$\mathbf{K}_6$
	$A_6.2$	$\mathbf{K}_6$
	$A_1 + K_5$	$\mathbf{K}_6$
	$D_6^{(3)}$	$\mathbf{K}_6$
	D_6	$\mathbf{K}_6$
	E_6	$\mathbf{K}_6$

Table 15. Reflection subgroup classes of $G_{35} = E_6$

	Class	Simple extensions
$\wp$	A_1	$A_2,\ 2A_1$
$\wp$	$2A_1$	$A_1 + A_2,\ A_3,\ 3A_1$
$\wp$	A_2	$A_1 + A_2,\ A_3$
$\wp$	$A_1 + A_2$	$2A_1 + A_2,\ A_1 + A_3,\ A_4,\ 2A_2$
$\wp$	A_3	$D_4,\ A_1 + A_3,\ A_4$
$\wp$	$3A_1$	$D_4,\ A_1 + A_3,\ 2A_1 + A_2,\ 4A_1$
	$4A_1$	$\mathbf{D}_4,\ 2A_1 + A_3$
$\wp$	$2A_1 + A_2$	$D_5,\ A_1 + A_4,\ 2A_1 + A_3,\ A_1 + 2A_2$
$\wp$	$A_1 + A_3$	$D_5,\ A_1 + A_4,\ 2A_1 + A_3,\ A_5$
$\wp$	A_4	$D_5,\ A_1 + A_4,\ A_5$
$\wp$	$2A_2$	$A_1 + 2A_2,\ A_5$
$\wp$	D_4	D_5
	$2A_1 + A_3$	$\mathbf{D}_5,\ A_1 + A_5$
$\wp$	$A_1 + 2A_2$	$E_6,\ A_1 + A_5,\ 3A_2$
$\wp$	$A_1 + A_4$	$E_6,\ A_1 + A_5$
$\wp$	A_5	$E_6,\ A_1 + A_5$
$\wp$	D_5	E_6
	$A_1 + A_5$	$\mathbf{E}_6$
	$3A_2$	$\mathbf{E}_6$

Table 16. Reflection subgroup classes of $G_{36} = E_7$

	Class	Simple extensions (ranks 1 to 5)
$\wp$	A_1	$2A_1,\ A_2$
$\wp$	$2A_1$	$A_1 + A_2,\ A_3,\ 3A_1.1,\ 3A_1.2$
$\wp$	A_2	$A_3,\ A_1 + A_2$
$\wp$	$A_1 + A_2$	$(A_1 + A_3).1,\ (A_1 + A_3).2,\ 2A_1 + A_2,\ A_4,\ 2A_2$
$\wp$	A_3	$D_4,\ (A_1 + A_3).1,\ (A_1 + A_3).2,\ A_4$
$\wp$	$3A_1.1$	$D_4,\ 2A_1 + A_2,\ (A_1 + A_3).1,\ 4A_1.1,\ 4A_1.2$
$\wp$	$3A_1.2$	$(A_1 + A_3).2,\ 4A_1.2$
	$4A_1.1$	$\mathbf{D}_4,\ (2A_1 + A_3).1,\ 5A_1$
$\wp$	$4A_1.2$	$A_1 + D_4,\ (2A_1 + A_3).2,\ 3A_1 + A_2,\ 5A_1$
$\wp$	$(A_1 + A_3).1$	$D_5,\ A_1 + D_4,\ A_1 + A_4,\ A_2 + A_3,$ $(2A_1 + A_3).1,\ (2A_1 + A_3).2,\ A_5.1$
$\wp$	$(A_1 + A_3).2$	$A_1 + D_4,\ (2A_1 + A_3).2,\ A_5.2$
$\wp$	$2A_1 + A_2$	$D_5,\ A_1 + A_4,\ A_1 + 2A_2,\ A_2 + A_3,\ 3A_1 + A_2,$ $(2A_1 + A_3).1,\ (2A_1 + A_3).2$
$\wp$	A_4	$D_5,\ A_1 + A_4,\ A_5.1,\ A_5.2$
$\wp$	$2A_2$	$A_2 + A_3,\ A_1 + 2A_2,\ A_5.1,\ A_5.2$
$\wp$	D_4	$D_5,\ A_1 + D_4$
	$5A_1$	$\mathbf{A}_1 + \mathbf{D}_4,\ 2A_1 + D_4,\ 3A_1 + A_3,\ 6A_1$
	$(2A_1 + A_3).1$	$\mathbf{D}_5,\ 2A_1 + D_4,\ 3A_1 + A_3,\ (A_1 + A_5).1,\ 2A_3$
$\wp$	$(2A_1 + A_3).2$	$D_6,\ A_1 + D_5,\ 2A_1 + D_4,\ (A_1 + A_5).2,$ $A_1 + A_2 + A_3,\ 3A_1 + A_3$
$\wp$	$A_1 + D_4$	$D_6,\ A_1 + D_5,\ 2A_1 + D_4$
$\wp$	$A_1 + A_4$	$E_6,\ A_2 + A_4,\ A_1 + D_5,\ A_6,\ (A_1 + A_5).1,\ (A_1 + A_5).2$
$\wp$	$A_2 + A_3$	$D_6,\ A_6,\ A_1 + A_2 + A_3,\ A_2 + A_4,\ 2A_3$
$\wp$	$A_1 + 2A_2$	$E_6,\ A_1 + A_2 + A_3,\ (A_1 + A_5).1,\ (A_1 + A_5).2,\ A_2 + A_4,\ 3A_2$
$\wp$	$3A_1 + A_2$	$A_1 + D_5,\ 3A_1 + A_3,\ A_1 + A_2 + A_3$
$\wp$	$A_5.1$	$E_6,\ A_6,\ D_6,\ (A_1 + A_5).1$
$\wp$	$A_5.2$	$D_6,\ (A_1 + A_5).2$
$\wp$	D_5	$E_6,\ D_6,\ A_1 + D_5$

TABLE 17. Reflection subgroup classes of $G_{36} = E_7$ (continued)

	Class	Simple extensions (ranks 6 and 7)
	$6A_1$	$2A_1 + D_4,\ 3A_1 + D_4,\ 7A_1$
	$2A_1 + D_4$	$\mathbf{D}_6,\ A_1 + D_6,\ 3A_1 + D_4$
	$2A_3$	$\mathbf{D}_6,\ A_7,\ A_1 + 2A_3$
	$3A_1 + A_3$	$\mathbf{A}_1 + \mathbf{D}_5,\ A_1 + D_6,\ A_1 + 2A_3,\ 3A_1 + D_4$
	$3A_2$	$\mathbf{E}_6,\ A_2 + A_5$
	$(A_1 + A_5).1$	$\mathbf{E}_6,\ A_1 + D_6,\ A_7$
$\wp$	$(A_1 + A_5).2$	$E_7,\ A_2 + A_5,\ A_1 + D_6$
$\wp$	$A_1 + D_5$	$E_7,\ A_1 + D_6$
$\wp$	$A_2 + A_4$	$E_7,\ A_2 + A_5,\ A_7$
$\wp$	$A_1 + A_2 + A_3$	$E_7,\ A_1 + D_6,\ A_2 + A_5,\ A_1 + 2A_3$
$\wp$	A_6	$E_7,\ A_7$
$\wp$	D_6	$E_7,\ A_1 + D_6$
$\wp$	E_6	E_7
	$7A_1$	$3A_1 + D_4$
	$3A_1 + D_4$	$A_1 + D_6$
	$A_1 + 2A_3$	$\mathbf{E}_7,\ A_1 + D_6$
	$A_1 + D_6$	$\mathbf{E}_7$
	$A_2 + A_5$	$\mathbf{E}_7$
	A_7	$\mathbf{E}_7$

TABLE 18. Reflection subgroup classes of $G_{37} = E_8$

	Class	Simple extensions (ranks 1 to 4)
$\wp$	A_1	$2A_1,\ A_2$
$\wp$	$2A_1$	$A_1 + A_2,\ 3A_1,\ A_3$
$\wp$	A_2	$A_1 + A_2,\ A_3$
$\wp$	$A_1 + A_2$	$2A_1 + A_2,\ 2A_2,\ A_4,\ A_1 + A_3$
$\wp$	A_3	$D_4,\ A_4,\ A_1 + A_3$
$\wp$	$3A_1$	$D_4,\ 2A_1 + A_2,\ A_1 + A_3,\ 4A_1.1,\ 4A_1.2$
	$4A_1.1$	$\mathbf{D}_4,\ (2A_1 + A_3).1,\ 5A_1$
$\wp$	$4A_1.2$	$A_1 + D_4,\ (2A_1 + A_3).2,\ 3A_1 + A_2,\ 5A_1$
$\wp$	$2A_1 + A_2$	$D_5,\ A_1 + A_4,\ A_1 + 2A_2,\ A_2 + A_3,\ 3A_1 + A_2,$ $(2A_1 + A_3).1,\ (2A_1 + A_3).2$
$\wp$	$2A_2$	$A_2 + A_3,\ A_5,\ A_1 + 2A_2$
$\wp$	A_4	$D_5,\ A_1 + A_4,\ A_5$
$\wp$	$A_1 + A_3$	$D_5,\ A_1 + D_4,\ A_1 + A_4,\ A_2 + A_3,\ A_5,$ $(2A_1 + A_3).1,\ (2A_1 + A_3).2$
$\wp$	D_4	$D_5,\ A_1 + D_4$

Table 19. Reflection subgroup classes of $G_{37} = E_8$ (continued)

	Class	Simple extensions (ranks 5 and 6)
	$5A_1$	$\mathbf{A}_1 + \mathbf{D}_4,\ 3A_1 + A_3,\ 2A_1 + D_4,\ 4A_1 + A_2,\ 6A_1$
	$(2A_1 + A_3).1$	$\mathbf{D}_5,\ (A_1 + A_5).1,\ 2A_3.1,\ 2A_1 + D_4,\ 3A_1 + A_3$
$\wp$	$(2A_1 + A_3).2$	$D_6,\ (A_1 + A_5).2,\ 2A_3.2,\ 2A_1 + D_4,\ 3A_1 + A_3,$ $A_1 + D_5,\ 2A_1 + A_4,\ A_1 + A_2 + A_3$
$\wp$	$A_1 + A_4$	$E_6,\ A_1 + D_5,\ (A_1 + A_5).1,\ (A_1 + A_5).2,\ A_2 + A_4,$ $2A_1 + A_4,\ A_6$
$\wp$	$A_1 + 2A_2$	$E_6,\ (A_1 + A_5).1,\ (A_1 + A_5).2,\ A_2 + A_4,\ A_1 + A_2 + A_3,$ $2A_1 + 2A_2,\ 3A_2$
$\wp$	$A_2 + A_3$	$D_6,\ A_2 + D_4,\ A_6,\ A_2 + A_4,\ 2A_3.1,\ 2A_3.2,\ A_1 + A_2 + A_3$
$\wp$	$3A_1 + A_2$	$A_1 + D_5,\ A_2 + D_4,\ A_1 + A_2 + A_3,\ 2A_1 + A_4,\ 3A_1 + A_3,$ $2A_1 + 2A_2,\ 4A_1 + A_2$
$\wp$	D_5	$E_6,\ D_6,\ A_1 + D_5$
$\wp$	A_5	$E_6,\ D_6,\ A_6,\ (A_1 + A_5).1,\ (A_1 + A_5).2$
$\wp$	$A_1 + D_4$	$D_6,\ A_2 + D_4,\ A_1 + D_5,\ 2A_1 + D_4$
	$6A_1$	$2A_1 + D_4,\ 3A_1 + D_4,\ 4A_1 + A_3,\ 7A_1$
	$2A_1 + D_4$	$\mathbf{D}_6,\ A_1 + D_6,\ A_3 + D_4,\ 2A_1 + D_5,\ 3A_1 + D_4$
	$3A_1 + A_3$	$\mathbf{A}_1 + \mathbf{D}_5,\ 2A_1 + A_5,\ A_1 + D_6,\ 2A_1 + D_5,\ A_3 + D_4,$ $A_1 + 2A_3,\ 3A_1 + D_4,\ 2A_1 + A_2 + A_3,\ 4A_1 + A_3$
	$4A_1 + A_2$	$\mathbf{A}_2 + \mathbf{D}_4,\ 2A_1 + D_5,\ 2A_1 + A_2 + A_3,\ 4A_1 + A_3$
	$3A_2$	$\mathbf{E}_6,\ A_2 + A_5,\ A_1 + 3A_2$
	$2A_3.1$	$\mathbf{D}_6,\ A_7.1,\ A_3 + D_4,\ A_1 + 2A_3$
	$(A_1 + A_5).1$	$\mathbf{E}_6,\ A_7.1,\ A_1 + D_6,\ 2A_1 + A_5$
$\wp$	$2A_3.2$	$A_7.2,\ D_7,\ A_3 + A_4,\ A_3 + D_4$
$\wp$	$(A_1 + A_5).2$	$E_7,\ A_7.2,\ A_1 + D_6,\ 2A_1 + A_5,\ A_1 + E_6,\ A_1 + A_6,\ A_2 + A_5$
$\wp$	E_6	$E_7,\ A_1 + E_6$
$\wp$	D_6	$E_7,\ D_7,\ A_1 + D_6$
$\wp$	A_6	$E_7,\ D_7,\ A_7.1,\ A_7.2,\ A_1 + A_6$
$\wp$	$A_2 + A_4$	$E_7,\ A_7.1,\ A_7.2,\ A_3 + A_4,\ A_2 + D_5,\ A_2 + A_5,\ A_1 + A_2 + A_4$
$\wp$	$A_1 + D_5$	$E_7,\ A_1 + D_6,\ D_7,\ A_1 + E_6,\ 2A_1 + D_5,\ A_2 + D_5$
$\wp$	$A_1 + A_2 + A_3$	$E_7,\ A_1 + D_6,\ A_2 + D_5,\ A_3 + A_4,\ A_1 + A_6,\ 2A_1 + A_2 + A_3,$ $A_1 + A_2 + A_4,\ A_2 + A_5,\ A_1 + 2A_3$
$\wp$	$2A_1 + A_4$	$D_7,\ 2A_1 + A_5,\ A_1 + A_6,\ A_3 + A_4,\ 2A_1 + D_5,$ $A_1 + E_6,\ A_1 + A_2 + A_4$
$\wp$	$2A_1 + 2A_2$	$2A_1 + A_5,\ A_2 + D_5,\ A_1 + A_2 + A_4,\ A_1 + E_6,\ A_1 + 3A_2,$ $2A_1 + A_2 + A_3$
$\wp$	$A_2 + D_4$	$D_7,\ A_2 + D_5,\ A_3 + D_4$

TABLE 20. Reflection subgroup classes of $G_{37} = E_8$ (continued)

	Class	Simple extensions (ranks 7 and 8)
	$7A_1$	$3A_1 + D_4,\ 4A_1 + D_4,\ 8A_1$
	$4A_1 + A_3$	$2A_1 + D_5,\ A_3 + D_4,\ 2A_1 + D_6,\ 2A_1 + 2A_3,\ 4A_1 + D_4$
	$3A_1 + D_4$	$\mathbf{A}_1 + \mathbf{D}_6,\ 2A_1 + D_6,\ 2D_4,\ 4A_1 + D_4$
	$2A_1 + D_5$	$\mathbf{D}_7,\ A_1 + E_7,\ 2A_1 + D_6,\ A_3 + D_5$
	$A_3 + D_4$	$\mathbf{D}_7,\ D_8,\ A_3 + D_5,\ 2D_4$
	$2A_1 + A_2 + A_3$	$\mathbf{A}_2 + \mathbf{D}_5,\ 2A_1 + D_6,\ A_1 + E_7,\ A_3 + D_5,\ A_1 + A_2 + A_5,\ 2A_1 + 2A_3$
	$A_1 + 3A_2$	$\mathbf{A}_1 + \mathbf{E}_6,\ A_1 + A_2 + A_5,\ A_2 + E_6,\ 4A_2$
	$2A_1 + A_5$	$\mathbf{A}_1 + \mathbf{E}_6,\ D_8,\ A_1 + A_7,\ 2A_1 + D_6,\ A_1 + E_7,\ A_1 + A_2 + A_5$
	$A_1 + 2A_3$	$\mathbf{E}_7,\ A_1 + D_6,\ A_3 + D_5,\ A_1 + A_7,\ 2A_1 + 2A_3$
	$A_2 + A_5$	$\mathbf{E}_7,\ A_8,\ A_2 + E_6,\ A_1 + A_2 + A_5$
	$A_1 + D_6$	$\mathbf{E}_7,\ D_8,\ A_1 + E_7,\ 2A_1 + D_6$
	$A_7.1$	$\mathbf{E}_7,\ D_8,\ A_1 + A_7$
$\wp$	$A_7.2$	$E_8,\ D_8,\ A_8$
$\wp$	D_7	$E_8,\ D_8$
$\wp$	E_7	$E_8,\ A_1 + E_7$
$\wp$	$A_1 + A_6$	$E_8,\ A_1 + A_7,\ A_8,\ A_1 + E_7$
$\wp$	$A_1 + E_6$	$E_8,\ A_1 + E_7,\ A_2 + E_6$
$\wp$	$A_3 + A_4$	$E_8,\ D_8,\ A_8,\ A_3 + D_5,\ 2A_4$
$\wp$	$A_2 + D_5$	$E_8,\ D_8,\ A_2 + E_6,\ A_3 + D_5$
$\wp$	$A_1 + A_2 + A_4$	$E_8,\ A_1 + E_7,\ A_2 + E_6,\ A_1 + A_7,\ A_1 + A_2 + A_5,\ 2A_4$
	$2A_1 + 2A_3$	$2A_1 + D_6,\ A_1 + E_7,\ A_3 + D_5$
	$8A_1$	$4A_1 + D_4$
	$4A_1 + D_4$	$2D_4,\ 2A_1 + D_6$
	$2A_1 + D_6$	$D_8,\ A_1 + E_7$
	$4A_2$	$A_2 + E_6$
	$2D_4$	D_8
	$A_1 + A_2 + A_5$	$\mathbf{E}_8,\ A_1 + E_7,\ A_2 + E_6$
	$A_3 + D_5$	$\mathbf{E}_8,\ D_8$
	$A_1 + A_7$	$\mathbf{E}_8,\ A_1 + E_7$
	$2A_4$	$\mathbf{E}_8$
	$A_2 + E_6$	$\mathbf{E}_8$
	A_8	$\mathbf{E}_8$
	$A_1 + E_7$	$\mathbf{E}_8$
	D_8	$\mathbf{E}_8$

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